Naval Research Laboratory

Washington, DC 20375-5000

NRL Report 9177

On the Problem of Optimal Signal Detection in Discrete-Time, Correlated, Non-Gaussian Noise

K. J. SANGSTON AND B. H. CANTRELL

Target Characteristics Branch
Radar Division

February 23, 1989

Approved for public release; distribution unlimited.

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Report 9177	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory 6b. OFFICE SYMBOL *(If applicable)* Code 5340.1S	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS *(City, State, and ZIP Code)* Washington, DC 20375-5000	7b. ADDRESS *(City, State, and ZIP Code)*
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Office of Naval Research 8b. OFFICE SYMBOL *(If applicable)*	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS *(City, State, and ZIP Code)* Arlington, VA 22217-5000	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO.: 61153N PROJECT NO.: 021-05-43 TASK NO.: WORK UNIT ACCESSION NO.: DN480-006

11. TITLE *(Include Security Classification)*

On the Problem of Optimal Signal Detection in Discrete-time, Correlated, Non-Gaussian Noise

12. PERSONAL AUTHOR(S)

Sangston, K.J. and Cantrell, B. H.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT *(Year, Month, Day)*	15. PAGE COUNT
Interim	FROM ________ TO ______	1989 February 23	26

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
FIELD	GROUP	SUB-GROUP	Detection theory; Radar signal processing
			Non-Gaussian; Radar
			K distribution

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Recent results of the detection of signals in discrete-time, correlated, non-Gaussian noise in which the univariate statistics and a general covariance structure of the noise are known have been obtained. The results are predicated on the assumption that a solution to the signal detection problem based on knowledge of univariate statistics and a covariance structure is "reasonable," even though it is known that in general a non-Gaussian noise process is not completely specified by such information. To examine this issue of "reasonableness," we present two general non-Gaussian noise models that are equivalent in these assumed attributes and yet lead to fundamentally different detection structures. This difference in the detection structures indicates that the signal detection problem is not adequately formulated without additional knowledge of the structure of the non-Gaussian noise process. We further present a specific radar example to quantify the difference in the detection structures.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Kevin J. Sangston	22b. TELEPHONE *(Include Area Code)* (202) 767-3472 22c. OFFICE SYMBOL Code 5340.1S

DD Form 1473, JUN 86 *Previous editions are obsolete.* SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603

CONTENTS

ON THE PROBLEM OF OPTIMAL SIGNAL DETECTION IN DISCRETE-TIME, CORRELATED, NON-GAUSSIAN NOISE

INTRODUCTION

Optimal signal detection in an environment of non-Gaussian noise is an important and difficult problem, and solutions to it have been obtained only under limited conditions. The need to identify optimal detection structures is motivated theoretically by the observation that a failure to recognize noise as non-Gaussian can lead to significant detection performance degradation in cases such as radar signal detection scenarios.

Because of the need to identify optimal detection structures, most researchers in this area have proceeded in one of two ways. One approach has been to assume that the samples that make up the observed data vector are independent and identically distributed according to a given non-Gaussian probability density function (pdf) [1-13]. Since the problem of specifying the optimal signal detection structure requires knowledge of the multivariate pdf of the noise vector, this problem may be, at least theoretically, solved in its entirety, since the knowledge of the univariate pdf (i.e., the marginal pdf) is sufficient to fully determine the required multivariate pdf for independent samples.

A second approach has been to obtain the asymptotically optimal detection structure [14-25] for a given noise background. Although this problem is not limited to the assumption of independence, the resulting detection structures often require a large number of samples to achieve acceptable detection performance. The asymptotically optimal detection structure may achieve good performance in the small sample case (this result holds in the Gaussian case, for instance), but this issue is not easily addressed. In addition to these two prevalent approaches, a limited number of results have been obtained for the more general case of finite sample detection of signals in non-Gaussian noise with constraints imposed on the covariance structure of the noise process [26-28].

Recently, results have been obtained for the more general problem of detecting signals in non-Gaussian noise with a general covariance structure [29-31]. Those results were obtained by assuming that the univariate pdf and the covariance structure of the noise were known. A multivariate pdf with marginal pdfs equal to a given univariate pdf and with the assumed covariance structure was constructed, and the detection structure was determined by applying the Neyman-Pearson criterion to the resulting likelihood ratio. The approach seems reasonable, but it is not without some arbitrariness. Although the Gaussian random process is completely specified by its mean and covariance function, a non-Gaussian process is usually not completely specified by such information. One may then ask if knowledge of the univariate statistics and the covariance function of a non-Gaussian process is sufficient (or even reasonable) for solving the problem of optimum signal detection in this non-Gaussian noise.

To examine this issue, we present two general models (both previously described in the literature) for non-Gaussian noise and show that they may be used to construct two different non-Gaussian multivariate pdfs that have the same marginal pdf and the same covariance structure. We restrict our attention to the more general problem of detection in the presence of narrowband noise, although the

Manuscript approved July 25, 1988.

application of our results to the case of strictly real random variables is straightforward. We show that, in general, these two models lead to different optimal detection structures. We also present an example, applicable to the radar case, which shows that the detection performance differs significantly for the two models, even though they are equivalent in the major attributes we have indicated.

NON-GAUSSIAN, CORRELATED NOISE MODELS

In this section of the report, we present two narrowband non-Gaussian, correlated noise models. These models are constructed based on an assumed knowledge of a univariate amplitude pdf and a covariance structure. More specifically, let x_i be a complex noise sample

$$x_i = |x_i| \cos \phi_i + j\,|x_i| \sin \phi_i \quad i = 1, \ldots , m. \tag{1}$$

We assume that the pdf of $|x_i|$, $h(|x_i|)$, is known, and that the phase ϕ_i is independent of $|x_i|$ and is uniformly distributed between 0 and 2π. This assumption leads to

$$f_{x_i} = \frac{h(|x_i|)}{2\pi |x_i|} \quad i = 1, \ldots , m, \tag{2}$$

where f_{x_i} is the complex univariate pdf of x_i.

Note that the x_i are identically distributed. We assume also that a covariance matrix R_x is known, where

$$R_x = [E(x_i \bar{x}_j)], \quad i, j = 1, \ldots . m \tag{3}$$

and the bar indicates complex conjugate.

Transformation Noise Process

The first model is the so-called transformation noise model described for real variables in Ref. 24, and applied to the narrowband-signal case in Refs. 29 through 31. For the narrowband-signal case, the model may be formulated as follows.

Specify a one-to-one invertible, nonlinear transformation

$$|y_i| = g(|x_i|) \quad i = 1, \ldots , m \tag{4a}$$

$$|x_i| = g^{-1}(|y_i|) \quad i = 1, \ldots , m \tag{4b}$$

such that $|y_i|$ is a Rayleigh random variable, and $|x_i|$ is a random variable distributed according to $h(|x_i|)$ as specified above. In general, this nonlinear mapping is given by

$$|y_i| = g(|x_i|) = \sqrt{-2\sigma_y^2 \ln [1 - H(|x_i|)]}, \quad i = 1, \ldots , m \tag{5}$$

where

σ_y^2 is the Rayleigh parameter (same for all i),

$H(\,)$ is the Cumulative distribution function associated with the pdf $h(\,)$.

This mapping is then applied to the samples x_i, y_i as

$$y_i = V(x_i) = \frac{g(|x_i|)}{|x_i|} x_i \quad i = 1, \ldots, m \tag{6a}$$

$$x_i = U(y_i) = \frac{g^{-1}(|y_i|)}{|y_i|} y_i \quad i = 1, \ldots, m \tag{6b}$$

where

$y_i = [y_1 \ldots y_m]^T$ is the vector of complex samples with a zero mean multivariate Gaussian pdf

$x_i = [x_1 \ldots x_m]^T$ is the vector of complex samples with a non-Gaussian multivariate pdf

T stands for transpose.

Note that the mapping indicated in Eqs. (5), (6a), and (6b) transforms only the amplitude of the complex sample while leaving the phase unchanged. As a result, the vector x is a complex vector with complex marginal pdfs given by Eq. (2). The multivariate pdf of x may be obtained by the straightforward application of the theory of transformation of random variables and is given by

$$f_x(x) = \frac{|J|}{(2\pi)^m \, |R_y|} \exp\left\{-\frac{1}{2} V(x)^t R_y^{-1} V(x)\right\}, \tag{7}$$

where

$x = [x_1 \ldots x_m]^T$,

$V(x) = [V(x_1) \ldots V(x_m)]^T$,

R_y is the covariance matrix of underlying complex Gaussian random vector y,

J is the Jacobian matrix associated with the transformation

$|\;|$ is the matrix determinant, and

t is the complex conjugate transpose.

As shown in Ref. 29, the term $|J|$ is in general equal to

$$|J| = \prod_{i=1}^{m} \frac{g(|x_i|)\dfrac{dg(|x_i|)}{d|x_i|}}{|x_i|}, \tag{8a}$$

which in this case may easily be shown to be equal to

$$|J| = \prod_{i=1}^{m} \frac{h(|x_i|)\sigma_y^2}{|x_i|(1 - H(|x_i|))}. \tag{8b}$$

To complete the discussion of the model, we must examine the relationship between R_y, the covariance matrix of y and R_x, the covariance matrix of x, which we have assumed is known. At this point, it suffices to say that since the nonlinear transformation given by Eqs. (5), (6a), and (6b) is a one-to-one mapping, the relationship between R_y and R_x is unique. We may therefore construct any R_x by the appropriate choice of R_y. Further discussion of this point is deferred to the presentation of an example.

To summarize, a complex non-Gaussian multivariate pdf,

$$f_x(x) = \frac{1}{(2\pi)^m |R_y|} \left[\prod_{i=1}^{m} \frac{h(|x_i|)\sigma_y^2}{|x_i|(1 - H(|x_i|))} \right] \exp\left\{ -\frac{1}{2} V(x)^t R_y^{-1} V(x) \right\},$$

may be constructed with a specified marginal pdf

$$f_{x_i}(x_i) = \frac{h(|x_i|)}{2\pi |x_i|} \quad i = 1, \ldots, m,$$

and a specified covariance matrix R_x that is uniquely determined by the choice of R_y.

Spherically Invariant Random Noise Process

A second non-Gaussian, correlated noise model may be formulated based on the class of spherically invariant random processes described in Refs. 32 through 35 and applied to the radar sea clutter modeling problem in Ref. 36. For the narrowband signal case, the model may be applied as follows.

On any given observation, the observed noise vector is a vector sample from a Gaussian distribution with a known normalized correlation matrix but with an unknown variance that is the same for all the samples within the observed vector. The unknown variance is then assumed to be a random variable. Conditioned on this variance, σ_x^2, the multivariate pdf of x, is

$$f_{x|\sigma_x^2}(x \mid \sigma_x^2) = \frac{1}{(2\pi)^m |\Lambda| (\sigma_x^2)^m} \, exp \left\{ -\frac{1}{2\sigma_x^2} x^t \Lambda^{-1} x \right\}, \tag{9}$$

where

$x = [x_1 \ldots x_m]^T$ is the vector of complex samples with a conditional Gaussian multivariate pdf,

Λ is the normalized correlation matrix of x,

$\sigma_x{}^2$ is the variance (i.e., power level) of x given observation

and is a random variable that fluctuates from one observation to the next.

The complex multivariate pdf of x is then given by

$$f_x(x) = \frac{1}{(2\pi)^m \, |\Lambda|} \int_0^\infty \frac{1}{(\sigma_x{}^2)^m} \exp\left\{\frac{-1}{2\sigma_x{}^2} x^t \Lambda^{-1} x\right\} p(\sigma_x{}^2) d(\sigma_x{}^2), \tag{10}$$

where

$p(\sigma_x{}^2)$ is the pdf of the random variable $\sigma_x{}^2$.

To ensure that $f_x(x)$ has the assumed marginal pdf given by Eq. (2), it is sufficient to choose $p(\sigma_x{}^2)$ as the solution to the following integral equation:

$$h(|x_i|) = \int_0^\infty \frac{|x_i|}{\sigma_x{}^2} \exp\left\{-\frac{|x_i|^2}{2\sigma_x{}^2}\right\} p(\sigma_x{}^2) d(\sigma_x{}^2), \tag{11}$$

or equivalently to show that the random variable $|x_i|$ (whose statistics we have assumed are known) has the following decomposition:

$$|x_i| = \sigma_x R_i \quad i = 1, \ldots, m, \tag{12}$$

where

R_i = the Rayleigh random variable with parameter = 1

$\sigma_x = \sqrt{\sigma_x^2}.$

The covariance matrix of x, R_x, may be shown in a straightforward manner to be given by

$$R_x = \mu_{\sigma_x{}^2} \Lambda, \tag{13}$$

where

$$\mu_{\sigma_x^2} = E[\sigma_x^2]$$

and is the mean value of σ_x^2.

To summarize, a complex non-Gaussian multivariate pdf,

$$f_x(x) = \frac{1}{(2\pi)^m \mid \Lambda \mid} \int_0^\infty \frac{1}{(\sigma_x^2)^m} \exp\left\{-\frac{1}{2\sigma_x^2} x^t \Lambda^{-1} x\right\} p(\sigma_x^2) d(\sigma_x^2),$$

may be constructed with a specified marginal pdf

$$f_{x_i}(x_i) = \frac{h(|x_i|)}{2\pi |x_i|} \qquad i = 1, \ldots, m,$$

and a specified covariance matrix R_x that is uniquely determined by the choice of Λ.

At this point, we have presented two different multivariate non-Gaussian pdfs, each with the same marginal pdf and the same specified covariance matrix. The question is if these two different noise models that are equivalent in major attributes, i.e., marginal pdf and covariance matrix, lead to *similar* detection structures. This question is tantamount to asking if the specification of the marginal pdfs and of the covariance structure of a noise process is sufficient to solve the signal detection problem in non-Gaussian noise.

OPTIMAL DETECTION STRUCTURES

The detection problem of interest is a binary hypothesis test

$$H_0: x = n$$

$$H_1: x = s + n,$$

where

n is the non-Gaussian noise vector,

s is the known signal vector.

The optimality criterion applied here is the Neyman-Pearson criterion. The optimal detection structure is therefore given by a likelihood ratio test

$$\lambda(x) = \frac{f_x(x \mid H_1)}{f_x(x \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{14}$$

where

$f_x(x \mid H_1)$ is the pdf of the observed vector x under H_1

$f_x(x \mid H_0)$ is the pdf of the observed vector x under H_0

T is the threshold chosen as a function of P_{fa}.

For the transformation noise model we have

$$\lambda_{TN}(x) = \left[\prod_{i=1}^{m} A_i\right] \exp\left\{-\frac{1}{2}V(x-s)^t R_y^{-1} V(x-s) - V(x)^t R_y^{-1} V(x)\right\} \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{15}$$

where

$$A_i = \frac{\dfrac{h(|x_i - s_i|)\sigma_y^2}{|x_i - s_i|[1 - H(|x_i - s_i|)]}}{\dfrac{h(|x_i|)\sigma_y^2}{|x_i|[1 - H(|x_i|)]}}.$$

If $R_y = \sigma_y^2 I$, then after some algebra we have

$$\lambda_{TN}(x) = \prod_{i=1}^{m} \frac{\dfrac{h(|x_i - s_i|)}{|x_i - s_i|}}{\dfrac{h(|x_i|)}{|x_i|}}, \tag{16}$$

which is equivalent to

$$\lambda_{TN}(x) = \prod_{i=1}^{m} \frac{f_{x_i - s_i}(x_i - s_i)}{f_{x_i}(x_i)}, \tag{17}$$

i.e., the independent sample detector. Thus, when $R_y = \sigma_y^2 I$, we also have $R_x = \sigma_x^2 I$.

For the spherically invariant noise model, we have

$$\lambda_{SI}(x) = \frac{\displaystyle\int_0^\infty \frac{1}{(\sigma_x^2)^m} \exp\left\{\frac{-1}{2\sigma_y^2}(x-s)^t \Lambda^{-1}(x-s)\right\} p(\sigma_y^2) d(\sigma_y^2)}{\displaystyle\int_0^\infty \frac{1}{(\sigma_x^2)^m} \exp\left\{-\frac{1}{2\sigma_y^2} x^t \Lambda^{-1} x\right\} p(\sigma_y^2) d(\sigma_y^2)}. \tag{18}$$

We note immediately that the choice $R_x = \mu_{\sigma_x^2} I$ does not cause Eq. (18) to reduce to a form that is equivalent to Eq. (17). From this observation, we deduce that the two noise models, although equivalent in the major attributes discussed above, are fundamentally different in at least one respect. When the correlation between samples is zero, one model yields independent samples, whereas the other model does not.

However, this observation does not necessarily indicate that the two models may not be considered equivalent within some detection performance criterion. To examine this issue, we must further examine the detection structures. From our assumption that s is a completely known signal vector, the detection structure for the transformation noise model (Eq. (15)) is based on a binary hypothesis test comprised of simple hypotheses. The detection structure for the spherically invariant noise model, on the other hand, is seen from Eq. (18) to be a Bayesian implementation of a test comprised of composite hypotheses. The implication is that the test implemented in Eq. (15) is a most powerful test for the signal vector s, whereas the test implemented in Eq. (18) is not necessarily a most powerful test. In fact, since the most powerful test for the spherically invariant noise model is the matched filter test for Gaussian noise when the power level, σ_x^2, is known on each detection decision, no test is uniformly (i.e., uniform over all possible values of σ_x^2) most powerful for testing for s. From these observations, we conclude that the optimal detection structures for the two noise models are significantly different.

To answer the question about the adequacy of the a priori knowledge that we have assumed (i.e., the univariate pdf and the covariance function) for formulating a signal detector, we must examine the performance of the detector that results from a particular noise model against the data from the other model. Since the aim of these studies is to build a practical signal detector, we present two suboptimal but practical detectors and use them for the performance comparison.

SUBOPTIMAL DETECTION STRUCTURES

Since the rationale for deriving these suboptimal detectors from the optimal detectors is given elsewhere (in Ref. 29 for the transformation noise model, and in Ref. 37 for the spherically invariant noise model), these detectors will be presented here but not derived.

For the transformation noise model, the test is given by

$$V(\hat{s})^t R_x^{-1} V(x) \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{19}$$

where

$\hat{s}$ is the steering vector obtained from

$s = \alpha e^{j\phi} \hat{s}$,

α is the amplitude

ϕ is the phase.

For the spherically invariant noise model, the test is comprised of two parts and is given by

(1)
$$\text{If } \frac{|\hat{s}^t \Lambda^{-1} x|}{\hat{s}^t \Lambda^{-1} \hat{s}} > T_\alpha \tag{20a}$$

is true, then a signal detection is declared. If it is not true, then

(2)
$$\ln \left[\frac{\int_0^\infty \frac{1}{(\sigma_x{}^2)^m} \exp\left\{ -\frac{1}{2\sigma_x{}^2} \left| x^t \Lambda^{-1} x - \frac{|\hat{s}^t \Lambda^{-1} x|^2}{\hat{s}^t \Lambda^{-1} \hat{s}} \right| \right\} p(\sigma_x{}^2) d(\sigma_x{}^2)}{\int_0^\infty \frac{1}{(\sigma_x{}^2)^m} \exp\left\{ -\frac{1}{2\sigma_x{}^2} x^t \Lambda^{-1} x \right\} p(\sigma_x{}^2) d(\sigma_x{}^2)} \right] \underset{H_0}{\overset{H_1}{\gtrless}} T \tag{20b}$$

is evaluated.

Note that both tests are independent of the signal amplitude α and the signal initial phase ϕ, which often are unknown. The assumption of knowledge about $\hat{s}$, which represents a pulse-to-pulse phase shift, is justified, since a bank of such detectors, each matched to a different $\hat{s}$, may be implemented.

EXAMPLE

The Weibull pdf is of significant interest in the radar detection problem, since certain types of radar clutter have been shown to have amplitude statistics that are described by it. In terms of our earlier notation, we may write

$$h(|x_i|) = a\frac{\ln 2}{m^b} |x_i|^{a-1} \exp\left\{ -\frac{\ln 2}{m^b} |x_i|^a \right\} \quad i = 1, \ldots, m, \tag{21}$$

where

a is the skewness parameter

b is the median value of $|x_i|$.

The range $0.5 \leq a \leq 2.0$ is the usual range of interest in the radar case [38]. Note that the value $a = 2.0$ yields a Rayleigh distribution. By applying the results given above, the complex bivariate pdf is given in the transformation noise model by

$$f_x(x) = \frac{\left[\frac{\alpha \ln 2}{b^a} \right]^2}{(2\pi)^2 |R_y|} \exp\left\{ -\frac{1}{2} V(x)^t R_x{}^{-1} V(x) \right\}, \tag{22}$$

where

$$V(x) = [V(x_1) \;\; V(x_2)]^T,$$

$$V(x_i) = \frac{\sqrt{2 \ln 2}}{|x_i|} \left[\frac{|x_i|}{b} \right]^{\frac{a}{2}} \quad i = 1, 2.$$

We further let $R_y = \begin{bmatrix} 1 & \rho_y \\ \rho_y & 1 \end{bmatrix}$ and $b = 1$. Cantrell [29] and Farina [30] have shown that this choice leads to

$$R_x = \sigma^2 \begin{bmatrix} 1 & \rho_x \\ \rho_x & 1 \end{bmatrix}, \tag{23a}$$

where

$$\sigma^2 = \frac{\Gamma\left(\frac{2}{a}\right)}{a(\ln 2)^{2/a}} \tag{23b}$$

$$\rho_x = \frac{a}{2}\rho_y(1 - \rho_y{}^2)^{\left(\frac{2}{a}+1\right)} \frac{\Gamma\left(\frac{1}{a} + \frac{3}{a}\right)}{\Gamma\left(\frac{2}{a}\right)} F\left(\frac{1}{a} + \frac{3}{2}; \frac{1}{a} + \frac{1}{2}; 2; \rho_y{}^2\right), \tag{23c}$$

where

$\Gamma(\;)$ is the gamma function and

$F(\;;\;;\;)$ is the Gauss hypergeometric function.

To show that the Weibull amplitude pdf may fit into the framework of the spherically invariant noise model, we must show that the Weibull pdf satisfies the integral equation given in Eq. (11) or the decomposition given by Eq. (12). These properties are shown in Appendix A for skewness parameter $a = 2$, 1 and 1/2, whereas Kim [39] has given an approximate solution to the integral equation in Eq. (11) that is accurate for the range $1 \le a < 2$. As shown in Appendix B, the complex bivariate pdf for this case is given by

$$f_x(x) = \frac{\left(\frac{a \ln 2}{b^a}\right)^2}{(2\pi)^2 |\Lambda|} \frac{\exp\left\{-\frac{\ln 2}{b^a}(x^t \Lambda^{-1} x)^{a/2}\right\}}{(x^t \Lambda^{-1} x)^{2-a}} \left[1 + \frac{(2 - a)}{\frac{a \ln 2}{b^a}(x^t \Lambda^{-1} x)^{a/2}}\right]. \tag{24}$$

In the example above, we choose

$$\Lambda = \begin{bmatrix} 1 & \rho_x \\ \rho_x & 1 \end{bmatrix}, \tag{25a}$$

$$\mu_{\sigma_x^2} = \sigma^2 = \frac{\Gamma\left(\frac{2}{a}\right)}{a(\ln 2)^{\frac{2}{a}}}. \tag{25b}$$

Note that we do not actually *choose* $\mu_{\sigma_x^2}$ at will, for it is determined by $p(\sigma_x^2)$. However, since the Weibull pdf may fit as a spherically invariant noise model and

$$\mu_{\sigma_x^2} = \sigma^2 = \frac{1}{2} E[x_i \bar{x}_i],$$

by definition, Eq. (25b) is always true.

In the radar problem, it seems more intuitive to call the spherically invariant noise process a spatial noise process, since the variation in σ_x^2 is thought to be induced by a spatial variation in the radar backscatter process [40]. Thus, from now on, the spherically invariant random noise will be referred to as spatial noise.

To proceed we assume that

$$\hat{s} = [1 + j0 \;\; 0 + j1]^T$$

and evaluate each of the two suboptimal noise detectors given above against each of the two noise models. Table 1 shows the covariance information about the noise used in the examples:

Table 1 — Covariance Function for Example

a	σ^2	ρ_x
0.5	51.985	0.958
1.0	2.081	0.975
1.5	0.970	0.979
2.0	0.721	0.98

For the transformation noise model, $\rho_y = 0.98$ was used for all values of a (this value of ρ_y generates the values of ρ_x given in the table).

Since the computation of the detection performance did not prove to be tractable in closed form, the performance was evaluated by Monte Carlo simulation. To accomplish this evaluation, the detector was first applied against data consisting of noise samples only, and the detection threshold that yielded a desired probability of false alarm was determined through the use of the importance sampling technique [41,42]. This threshold was then used in conjunction with the detector operated against data consisting of noise samples plus signal samples with amplitude α, random initial phase ϕ, and steering vector $\hat{s}$. The performance results obtained are presented in Figs. 1 through 4, where

$$\frac{S}{C} = 10 \log \left[\frac{\alpha^2}{2\sigma^2} \right] . \tag{26}$$

The figures indicate the general trend that the transformation noise detector, when operated against the transformation noise model, yields uniformly the best detection performance. The performance in this particular case is significantly better than the performance in other cases at the spikiest clutter condition examined (the clutter is said to get spikier as $a \rightarrow 0$). On the other hand, the trend is that at the spikier clutter conditions, the transformation noise detector operated against the spatial noise model yields the worst performance of the scenarios tested. Since we have made the two models equivalent in the attributes that we have assumed are known, we may conclude that our earlier observations about the difference in the detection structures is supported by the example we have evaluated. It seems that knowledge of the univariate pdf and the covariance function of the non-Gaussian noise process is not sufficient for the solution of the signal detection problem.

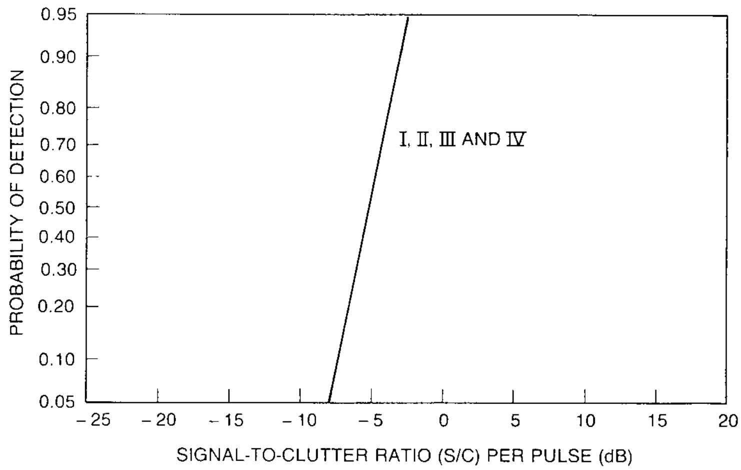

Fig. 1 — Performance, $\alpha = 2$, $\rho = 0.98$, $Pfa = 10e^{-6}$

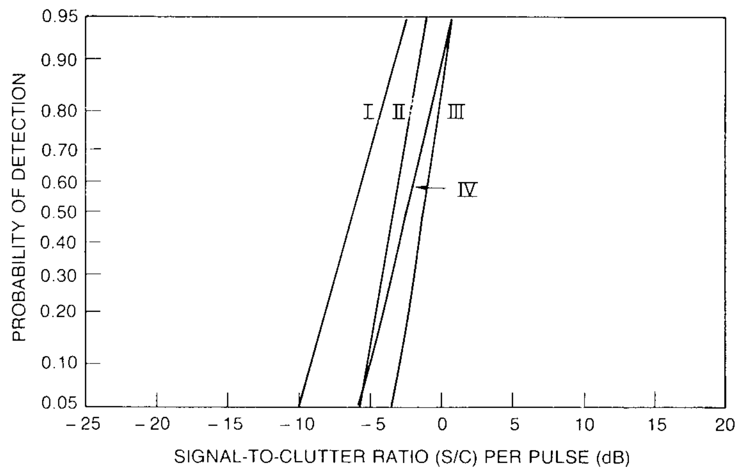

Fig. 2 — Performance, $\alpha = 1.5$, $\rho = 0.979$, $Pfa = 10e^{-7}$

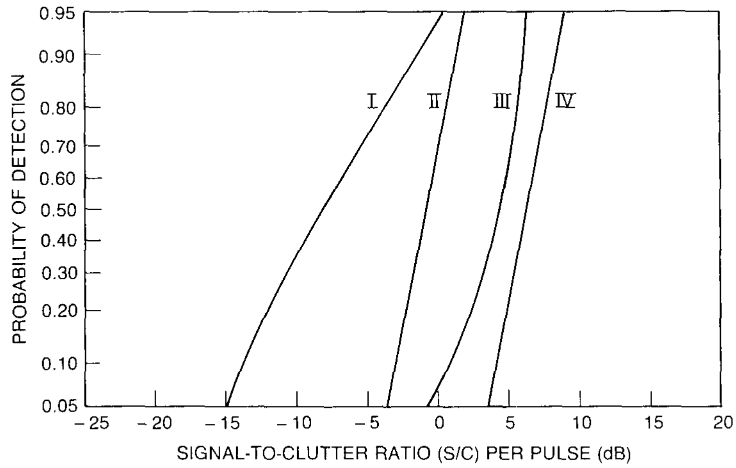

Fig. 3 — Performance, $\alpha = 1$, $\rho = 0.9751$, $Pfa = 10e^{-7}$

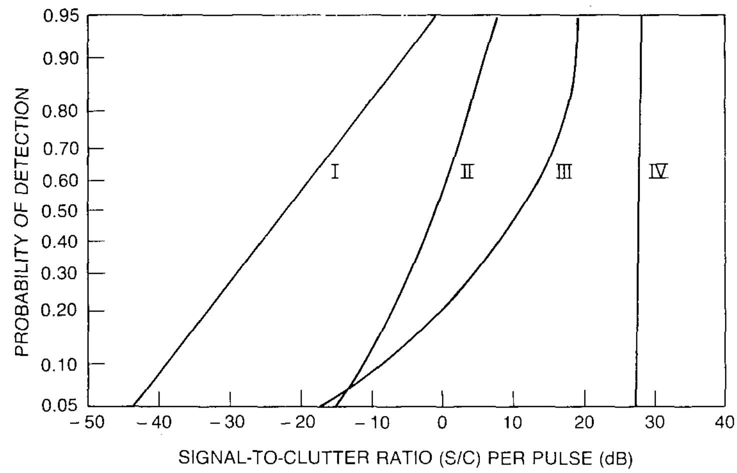

Fig. 4 — Performance, $\alpha = 0.5$, $\rho = 0.9582$, $Pfa = 10e^{-7}$

To give a further point of reference for the detection performance against the spikiest clutter, Fig. 4 is reproduced in Fig. 5 with the performance of the matched filter detector, $|\hat{s}^t \Lambda^{-1} x|$, against the two models added for comparison. This comparison indicates that in the absence of any additional a priori knowledge other than that which we have assumed, the matched filter detector is *safer* than the transformation noise detector if we seek to minimize the maximum risk that we take in implementing a detector. The spatial noise detector yields better performance than that of matched filter detector, but the increase in performance may not be sufficient to warrant the added complexity of implementing the spatial noise detector. A designer constrained by the complexity of implementation, unless the added complexity yields a significant payoff, may implement the matched filter rather than either of the two detectors presented here. Clearly, then, further research in this area should focus on determining what a priori information is sufficient to allow for an adequate formulation of the detection problem that realizes the potential gain in the detection performance that a non-Gaussian noise model offers relative to the Gaussian noise model.

SUMMARY

Two non-Gaussian noise models that may be made to have the same univariate statistics and the same covariance function are presented. The general optimal detection structure for each of the noise models with a completely known signal is given; the two detection structures are shown to differ in

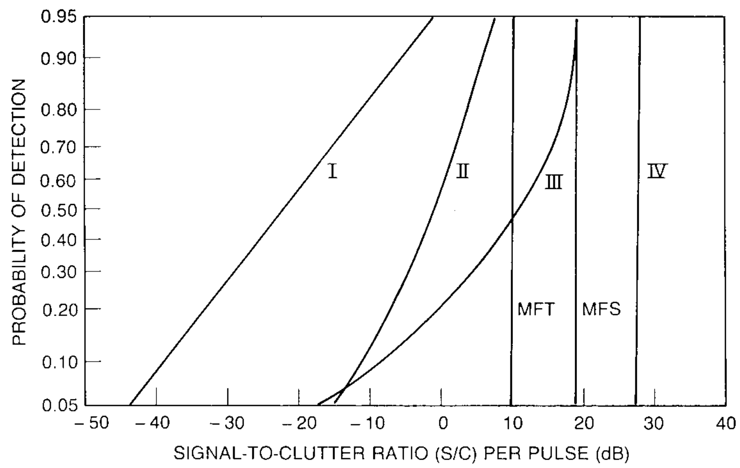

Fig. 5 — Performance, $\alpha = 0.5$, $\rho = 0.9582$, $Pfa = 10e^{-7}$

fundamental ways. This difference indicates that a noise model based on univariate statistics and a covariance function is not sufficient to formulate the solution of the signal detection problem in non-Gaussian noise. A radar example of significant practical interest is presented to support this general conclusion.

ACKNOWLEDGMENTS

The authors thank Dr. Karl Gerlach who pointed out the derivation presented in Appendix B, which is a simpler derivation than the one we had originally formulated.

REFERENCES

1. O.Ye. Antonov, "Optimum Detection of Signals in Non-Gaussian Noise," *Radio Eng. Elecron. Phys.* **12**, 541-548 (1967).

2. O.Ye. Antonov, "Optimum Detection of Signals in Non-Gaussian Noise. Detection of a Signal of Unknown Amplitude and Phase," *Radio Eng. Electron. Phys.* **12**, 727-735 (1967).

3. R.L. Spooner, "On the Detection of a Known Signal in a Non-Gaussian Noise Process," *J. Acous. Soc. Am.* **44**(1), 141-147. (1968).

4. C.P. Hatsell and L.W. Nolte, "On Transient and Periodic Signal Detection in Time-Varying Noise Power," *IEEE Trans. Aero. and Elec. Syst.* **AES-7**(6), 1100-1112 (1971).

5. J.H. Miller and J.B. Thomas, "Detectors for Discrete-Time Signals in Non-Gaussian Noise," *IEEE Trans. Inf. Theory* **IT-18**(2), 241-249 (1972).

6. J.H. Miller and J.B. Thomas, "The Detection of Signals in Impulsive Noise Modeled as a Mixture Process," *IEEE Trans. Comm.* **COM-24**(5), 559-563 (1976).

7. A.D. Spaulding and D. Middleton, "Optimum Reception in an Impulsive Interference Environment—Part I: Coherent Detection," *IEEE Trans. Comm.* **COM-25**(9), 910-923 (1977).

8. H.V. Poor and J.B. Thomas, "Locally Optimum Detection of Discrete-Time Stochastic Signals in Non-Gaussian Noise," *J. Acous. Soc. of Am.* **63**(1), 75-80 (1978).

9. J.J. Sheehy, "Optimum Detection of Signals in Non-Gaussian Noise," *J. Acous. Soc. Am.* **63**(1), 81-90 (1978).

10. R.J. Marks et al., "Detection in Laplace Noise," *IEEE Trans. Aero. Elec. Syst.* **AES-14**(6), 866-871 (1978).

11. J.W. Modestino and A.Y. Ningo, "Detection of Weak Signals in Narrowband Non-Gaussian Noise," *IEEE Trans. Inf. Theory* **IT-25**(5), 592-600 (1979).

12. E. Conte et al., "Asymptotically Optimum Radar Detection in Sea Clutter," MELECON '85 Conf. Procs. **2**, Digital Signal Processing, pp. 339-343.

13. S.A. Kassam, *Signal Detection in Non-Gaussian Noise (Springer-Verlag, New York, 1988).*

14. B. Picinbono and G. Vezzosi, "Detection d'un Signal Certain dans un Bruit Non Stationnaire et Non Gaussien," *Annales des Telecommunications* **25**, 433-439 (1970).

15. G. Vezzosi and B. Picinbono, "Detection d'un Signal Certain dans un Bruit Spheriquement Invariant et Caracteristiques des Recepteurs," *Annales des Telecommunications* **27** 95-110 (1972).

16. H.V. Poor and J.B. Thomas, "Memoryless Discrete-time Detection of a Constant Signal in m-Dependent Noise," *IEEE Trans. Inf. Theory* **IT-25**(1), 54-61 (1979).

17. D.R. Halverson and G.L. Wise, "Discrete-Time Detection in ϕ—Mixing Noise," *IEEE Trans. Inf. Theory* **IT-26**(2), 189-198 (1980).

18. D.R. Halverson and G.L. Wise, "A Detection Scheme for Dependent Noise Processes," *J. Franklin Inst.* **309**(5), 287-300 (1980).

19. D.R. Halverson and G.L. Wise, "Asymptotic Memoryless Detection of Random Signals in Dependent Noise," *J. Franklin Inst.* **312**, 13-29 (1981).

20. N.H. Lu and B.A. Eisenstein, "Detection of Weak Signals in Non-Gaussian Noise," *IEEE Trans. Inf. Theory* **IT-27**(6), 755-771 (1981).

21. N.H. Lu and B.A. Eisenstein, "Weak Signal Detection in Non-Gaussian Noise of Unknown Level," *IEEE Trans. Aero. Elect. Syst.* **AES-20**(6), 830-834 (1984).

22. D.R. Halverson and G.L. Wise, "Asymptotic Memoryless Discrete-Time Detection of ϕ—Mixing Signals in ϕ—Mixing Noise," *IEEE Trans. Inf. Theory* **IT-30**(2), 415-417 (1984).

23. D.R. Halverson and G.L. Wise, "Approximately Optimal Memoryless Detection of Random Signals in Dependent Noise," *IEEE Trans. Inf. Theory* **IT-30**(2), 420-424 (1984).

24. A.B. Martinez, P.F. Swaszek, and J.B. Thomas, "Locally Optimum Detection in Multivariate Non-Gaussian Noise," *IEEE Trans. Inf. Theory* **IT-30**(6), 815-822 (1984).

25. M.I. Dadi and R.J. Marks, "Detector Relative Efficiencies in the Presence of Laplace Noise," *IEEE Trans. Aero. Elect. Syst.* **AES-23**(4), 568-581 (1987).

26. R.L. Stratonovich and Yu. G. Sosulin, "Optimum Reception of Signals in Non-Gaussian Noise," *Radio Eng. Electron. Phys.* **11**, 497-506 (1966).

27. R.L. Stratonovich, "Detection and Estimation of Signals in Noise When One or Both are Non-Gaussian," *Proc. IEEE* **58**, 670-679 (1970).

28. H.V. Poor, "Signal Detection in the Presence of Weakly Dependent Noise—Part 1: Optimum Detection," *IEEE Trans. Inf. Theory* **IT-28**(5), 735-744 (1982).

29. B.H. Cantrell, "Radar Target Detection in Non-Gaussian Correlated Clutter," NRL Report 9015, Nov. 1986.

30. A. Farina et al., "Theory of Radar Detection in Coherent Weibull Clutter," *IEE Proc.* **134F**(2), 174-190 (1987).

31. B.H. Cantrell, "Detection of Signals in Non-Gaussian, Correlated Noise Derived from Cauchy Processes," NRL Report 9086, Oct. 1987.

32. A.M. Vershik, "Some Characteristic Properties of Gaussian Stochastic Processes," *Theory of Probability and Its Applications* **9**, 353-356 (1964).

33. I.F. Blake and J.B. Thomas, "On a Class of Processes Arising in Linear Estimation Theory," *IEEE Trans. Inf. Theory* **IT-14**(1), 12-16 (1968).

34. K. Yao, "A Representation Theorem and Its Application to Spherically Invariant Random Processes," *IEEE Trans. Inf. Theory* **IT-19**(5), 600-608 (1973).

35. B. Picinbono, "Spherically Invariant and Compound Gaussian Stochastic Processes," *IEEE Trans. Inf Theory* **IT-16**(1), 77-79 (1970).

36. E. Conte and M. Longo, "Characterization of Radar Clutter as a Spherically Invariant Random Process," *IEE Proc.* **134**, Part F(2), 191-197 (1987).

37. K.J. Sangston, "Coherent Detection of Radar Targets in K-Distributed, Correlated Clutter," NRL Report 9130, Aug. 1988.

38. M.I. Skolnik, *Introduction to Radar Systems*, 2nd ed. (McGraw-Hill Book Co., New York, 1980.

39. Kim, M.W., "Calculation of The Power Density in the Fredholm Equation that Yields a Weibull pdf", Naval Research Laboratory Report 9150, Washington, D.C. (September 14, 1988).

40. G.V. Trunk, "Radar Properties of Non-Rayleigh Sea Clutter", *IEEE Trans. Aero. Elect. Syst.* **AES-8**(2), 196-204 (1971).

41. R.L. Mitchell, "Importance Sampling Applied to Simulation of False Alarm Statistics," *IEEE Trans. Aero. Elect. Syst.* **AES-17**(1), 15-24 (1981).

42. A. diVito, G.G. Galati, and D. Iovino, "A Comparison of Variance Reduction Techniques for Radar Simulation," Radar-87, IEE Conf. **381**, 510-514 (1987).

Appendix A

DECOMPOSITION OF WEIBULL RANDOM VARIABLE FOR SELECTED VALUES OF THE SKEWNESS PARAMETER

The Weibull pdf is given by

$$h(\omega) = ab\,\omega^{a-1}e^{-b\omega^a} \quad \omega > 0, \tag{A-1}$$

where

a is the skewness parameter

b is the scale parameter.

We wish to show that for the values of the skewness parameter, $a = 2, 1,$ and $1/2$, the Weibull random variable ω has the decomposition

$$\omega = \sigma R \tag{A-2}$$

where

R is the Rayleigh random variable with unit parameter

σ is the random variable independent of R.

If we let $a = 2$, we have

$$h(\omega; a = 2) = 2b\,\omega e^{-b\omega^2}, \tag{A-3}$$

which is a Rayleigh pdf with parameter $1/2b$. Clearly, then, when $a = 2$, the random variable ω trivially satisfies the decomposition given by Eq. (A-2), i.e.,

$$\omega = \sqrt{\frac{1}{2b}}\,R. \tag{A-4}$$

For this value of the skewness parameter, the pdf of σ^2 is

$$p(\sigma^2) = \delta\left[\sigma^2 - \frac{1}{2b}\right], \tag{A-5}$$

where

$\delta(\ )$ is the Dirac delta function,

and the Weibull pdf may be written as

$$h(\omega;a = 2) = \int_0^{\infty} \frac{\omega}{\sigma^2} e^{-\frac{\omega^2}{2\sigma^2}} \delta\left[\sigma^2 - \frac{1}{2b}\right] d(\sigma^2). \tag{A-6}$$

If we let $a = 1$, we have for the Weibull pdf

$$h(\omega;a = 1) = be^{-b\omega}, \tag{A-7}$$

which is an exponential pdf. The exponential distribution is a special case of the K distribution, whose pdf is given by

$$f(\omega) = \frac{\frac{2\nu}{\eta}\omega}{\Gamma(\nu)} \left[\frac{\omega}{2}\sqrt{\frac{2\nu}{\eta}}\right]^{\nu-1} K_{\nu-1}\left[\omega\sqrt{\frac{2\nu}{\eta}}\right] \quad \omega,\nu,\eta > 0, \tag{A-8}$$

where

$\Gamma(\ )$ is the γ function,

$K_{\nu-1}(\ )$ is the modified Bessel function of the second kind of order $\nu - 1$

ν, η are the parameters of the distribution.

Since

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

$$K_{-\mu}(z) = K_{\mu}(z),$$

and

$$\Gamma(1/2) = \sqrt{\pi}, \tag{A-9}$$

then, if we let $\nu = 1/2$, we have

$$f(\omega) = \frac{\left[\frac{1}{\eta}\right]\omega}{\sqrt{\pi}} \sqrt{\frac{2}{\omega\sqrt{\frac{1}{\eta}}}} \sqrt{\frac{\pi}{2\omega\sqrt{\frac{1}{\eta}}}} e^{-\sqrt{\frac{1}{\eta}}\omega}$$

$$= \sqrt{\frac{1}{\eta}}\, e^{-\sqrt{\frac{1}{\eta}}\omega}. \tag{A-10}$$

which, if we set $\sqrt{1/\eta} = b$, is equivalent to our Weibull pdf with $a = 1$. The K pdf has the known functional decomposition

$$f(\omega) = \int_0^\infty \frac{\omega}{\sigma^2} e^{-\frac{\omega^2}{2\sigma^2}} p(\sigma^2) d(\sigma^2), \tag{A-11}$$

where

$$p(\sigma^2) = \frac{\left(\frac{\nu}{\eta}\right)^\nu}{\Gamma(\nu)} (\sigma^2)^{\nu-1} e^{-\frac{\nu}{\eta}\sigma^2}. \tag{A-12}$$

Thus, with $\nu = 1/2$ and $\eta = 1/b^2$, we have

$$h(\omega;a = 1) = \int_0^\infty \frac{\omega}{\sigma^2} e^{-\frac{\nu}{\eta}\sigma^2} \frac{b}{\sqrt{2\pi\sigma^2}} e^{-\frac{b^2}{2}\sigma^2} d(\sigma^2), \tag{A-13}$$

which corresponds to Eq. (A-2).

Finally to examine the case $a = 1/2$, we consider that any Weibull random variable may be shown to have the representation

$$\omega = (x)^{1/a} \tag{A-14}$$

where x is a random variable with pdf

$$p(x) = be^{-bx}, \qquad x \geq 0 \tag{A-15}$$

i.e., exponential. We have shown that the exponential random variable has the representation

$$x = \sqrt{\gamma} R, \tag{A-16}$$

where γ is a random variable with pdf

$$p(\gamma) = \frac{b}{\sqrt{2\pi\gamma}} e^{-\frac{b^2}{2}\gamma}. \tag{A-17}$$

Substituting of Eq. (A-16) into Eq. (A-14) shows that any Weibull random variable has the representation

$$\omega = (\sqrt{\gamma} R)^{1/a} \tag{A-18}$$

Let $t = R^{1/a}$ and examine the pdf of t. This pdf is easily shown to be

$$g(t) = \frac{2a}{2} t^{2a-1} e^{-t^{2a}/2}, \tag{A-19}$$

which shows that t is a Weibull random variable with skewness parameter $2a$ and scale parameter 1/2. But we have already shown in Eq. (A-18) that t may be represented as

$$t = (\sqrt{\beta} R)^{1/2a} \tag{A-20}$$

where the pdf of β is

$$p(\beta) = \frac{\left(\frac{1}{2}\right)}{\sqrt{2\pi\beta}} e^{-\beta/8} \tag{A-21}$$

(i.e., in Eq. (A-18), let $\omega \to t$, $a \to 2a$, $\gamma \to \beta$ and in Eq. (A-17) let $b \to 1/2$). Back substitution in Eq. (A-18) shows that a general Weibull random variable of skewness a and scale b may be represented as

$$\omega = (\sqrt{\gamma})^{1/a} (\sqrt{\beta} R)^{1/2a}. \tag{A-22}$$

At this point we could continue this process further, since we again have a random variable of the form R^p, where p is a constant, and we have shown that this random variable is a Weibull random variable. For our interest, though, we may stop with Eq. (A-22) and examine it when $a = 1/2$, i.e.,

$$\omega = \gamma \sqrt{\beta} R, \tag{A-23}$$

which is the decomposition we are seeking to show. For $a = 1/2$, we therefore have

$$h(\omega; a = 1/2) = \int_0^\infty \frac{\omega}{\sigma^2} e^{-\omega^2/2\sigma^2} p(\sigma^2) d(\sigma^2), \tag{A-24}$$

where

$$\sigma^2 = \gamma^2 \beta. \tag{A-25}$$

A straightforward application of the laws of transformation of random variables with Eqs. (A-17) and (A-21) yields

$$p(\sigma^2) = \frac{b}{4\pi\sqrt{\sigma^2}} \int_0^\infty \frac{1}{\gamma\sqrt{\gamma}} \exp\left\{\frac{\sigma^2}{8\gamma^2} - \frac{b^2\gamma}{2}\right\} d\gamma. \tag{A-26}$$

Further evaluation of this integral does not appear to be tractable.

Appendix B

MULTIVARIATE PDF FOR SPHERICALLY INVARIANT NOISE

Although more than one approach is available to derive the desired pdf, a very straightforward approach is as follows.

The multivariate pdf is given by

$$f_x(x) = \frac{1}{(2\pi)^m \, |\Lambda|} \int_0^\infty \frac{1}{(\sigma_x{}^2)^m} \exp\left\{-\frac{1}{2\sigma_x{}^2} x^t \Lambda^{-1} x\right\} p(\sigma_x{}^2) d(\sigma_x{}^2), \tag{B-1}$$

where $p(\sigma_x{}^2)$ satisfies

$$h(|x_i|) = \int_0^\infty \frac{|x_i|}{\sigma_x{}^2} \exp\left\{-\frac{|x_i|^2}{2\sigma_x{}^2}\right\} p(\sigma_x{}^2) d(\sigma_x{}^2), \tag{B-2}$$

and $h(|x_i|)$ is known. Define $z = 1/\sigma_x{}^2$. Substitution in Eqs. (B-1) and (B-2) yields

$$f_x(x) = \frac{1}{(2\pi)^m \, |\Lambda|} \int_0^\infty z^m \exp\left\{-\frac{z}{2} x^t \Lambda^{-1} x\right\} f(z) dz. \tag{B-3}$$

$$h(|x_i|) = \int_0^\infty |x_i| \exp\left\{-\frac{|x_i|^2}{2} z\right\} z f(z) dz. \tag{B-4}$$

Now define

$$q(z) = z f(z). \tag{B-5}$$

With this definition, Eq. (B-4) becomes

$$\frac{h(|x_i|)}{|x_i|} = \int_0^\infty q(z) \exp\left\{-\frac{|x_i|^2}{2} z\right\} dz. \tag{B-6}$$

If we let $s = |x_i|/2$ and $Q(s)$ be the one-sided Laplace Transform of $q(z)$, we then have

$$Q(s) = \frac{h(\sqrt{2s})}{\sqrt{2s}}, \tag{B-7}$$

and

$$q(z) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} Q(s) e^{sz} ds, \tag{B-8}$$

where c is greater than the real part of any singularities of Q.

We now use the result in Eqs. (B-5) and (B-8) in Eq. (B-3) to obtain

$$f_x(x) = \frac{1}{2\pi j} \frac{1}{(2\pi)^m |\Lambda|} \int_{c-j\infty}^{c+j\infty} \frac{h(\sqrt{2s})}{\sqrt{2s}} \int_0^\infty z^{m-1} \exp \left\{ - \left[\frac{1}{2} x^t \Lambda^{-1} x \right] z + zs \right\} dz ds. \tag{B-9}$$

Since $x^t \Lambda^{-1} x$ is positive definite, if we assume $m > 1$ (a trivial assumption) the integral over z converges, and we have

$$f_x(x) = \frac{(-1)^m}{2\pi j} \frac{(m-1)!}{(2\pi)^m |\Lambda|} \int_{c-j\infty}^{c+j\infty} \frac{h(\sqrt{2s})}{\sqrt{2s}} \frac{1}{\left[s - \frac{1}{2} x^t \Lambda^{-1} x \right]^m} ds. \tag{B-10}$$

If we now assume that $s = \sigma + j\omega$ and

$$\lim_{\sigma \to \infty} \frac{h(\sqrt{2s})}{\sqrt{2s}} \to 0,$$

and that $h(\sqrt{2s})/\sqrt{2s}$ is analytic in the right-half plane, then by complex analysis we have

$$f_x(x) = \frac{(-1)^{m-1}}{(2\pi)^m |\Lambda|} \frac{d^{m-1}}{ds^{m-1}} \left\{ \frac{h(\sqrt{2s})}{\sqrt{2s}} \right\}_{s = \frac{1}{2} x^t \Lambda^{-1} x}. \tag{B-11}$$

If $m = 2$ and $h(\,)$ is the Weibull pdf, the various assumptions that were made hold, and we get Eq. (24) of the main body of the report. Less straightforward approaches that do not use complex analysis yield the same result.